Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 15, Issue. 1, No.4 : 2024
ISSN : **1906-9685**

THE GEODETIC NUMBER IN THE COM PRODUCT OF GRAPHS

Mr M Gnanasekar Research Scholar Department of Mathematics, NallamuthuGounderMahalingam College, Pollachi, Tamil Nadu, India gnanasekar.kalam@gmail , **Dr S Sivasankar** Associate Professor Department of Mathematics NallamuthuGounderMahalingam College, Pollachi, Tamil Nadu, India sssankar@gmail.com.

Abstract
Let G and H be two simple, nontrivial and undirected graphs. Let o be a vertex of H, the comb product between G and H, denoted by $G \rhd H$, is a graph attained by taking one copy of G and $|V(G)|$ copies of H and grafting the i^{th} copy of H at the vertex o to the i^{th} vertex of G. A set of vertices S of a graph G is a geodetic set, if all the vertex of graph G lies in at least one interval between the vertices of S. The minimum cardinality of the geodetic set in G is the geodetic number of $g(G)$. In this paper, we determine an exact value of geodetic number in comb product of graphs.

1. Introduction

By a simple and connected graph $G = (V, E)$, we further assume that G has noisolated vertices. For graph theoretic terminology we refer to Chartrand and Lesniak [8].

The notions of distance in graphs is a well-studied topic with several practical applications. For any two vertices u and v of a connected graph G, the *distance*$d_G(u, v)$ is thelength of a shortest $u - v$ path in G. The *eccentricity* of a vertex u of a graph G is themaximum distance between u and any other vertex of G. The *diameter* of G, denoted by$diam(G)$, is the maximum eccentricity of vertices in G, and the *radius* is the minimum sucheccentricity. The *interval*$I_G[u, v]$ between u and v is the set of all vertices on all shortest$u - v$ paths. Given a set $S \subseteq V(G)$, its *geodetic closure*$I_G[S]$ is the set of all vertices lyingon some shortest path joining two vertices of S; that is,

$I_G[S] = \{v \in V(G) : v \in I_G[x, y], x, y \in S\} = \bigcup_{x,y\epsilon s} I_G[x, y]$

A set $S \subseteq V(G)$ is called a *geodetic set* in G if $I_G[S] = V(G)$; that is, every vertex in Glies on some geodesic between two vertices from S. The *geodetic number*$g(G)$ of a graph $G$is the minimum cardinality of a geodetic set in G.

In chemistry [1], some families of chemicalgraphs can be measured as the comb product graphs. Let G and H be two connected graphs.Let o be a vertex of H. The *comb product* between G and H, denoted by $G \rhd H$, is a graphobtained by taking one copy of G and $|V(G)|$ copies of H and identify the i^{th}copy of H atthe vertex o with the i^{th} vertex of G. By the definition of comb product, we can say that$V(G \rhd H) = (a, v)|a \in V(G), v \in V(H)$ and $(a,v)(b,w) \in E(G \rhd H)$ whenever $a = b$ and$vw \in E(H)$, or $ab \in E(G)$ and $v=w=o$. We consider two different vertices $a,b \in V(G)$and a vertex $o \in V(H)$. We define $H(a) = (a, v)|v \in V(H)$, $G(o) = (v, o)|v \in V(G)$, and$P_G(a, b)$ is the shortest path from a to b in G.

We find main results. The first result is related to $G \rhd H$ when G is a connectedgraph and H is the classes of graphs such as Path, Cycle, Completed Graph, Trees andWheel Graph. In the literature, the problem of the geodetic number of a graph was initiatedby Harary, Loukakis, andTsouros in 1986 and their result appeared as a published paper in1993[10]. We collect the basic definitions in graphs and geodetic sets which are used in thesubsequent chapters; for graph theoretic terminology we refer to the Chartrand and Lesniak[8] few basic results on the geodetic number of cartesian and strong product of graphs.These results are in Bresar et al. [3] and in Caceres et al. [5]

Lemma 1.1. [6] *Every geodetic set of a graph contains its extreme vertices*.

Lemma 1.2. [9] *Let deg : V(A) → V(B) be an isomorphism between graphs A and B.The set S is a geodetic set of A if and only if deg(x)|x ∈ S is a geodetic set of B*

Lemma 1.3. [9] *Let o ∈ V(H) be the identifying vertex and u, v be two distinct vertices of G ⊳ H. For l ∈ 1, 2, ..., n, if u ∈* V_l *and v ∉* V_l*, then every u − v path in G ⊳ Hconsists of* (g_l, o).

Lemma 1.4. [9] *Let o ∈ V (H) be the identifying vertex and S be a geodetic set of G ⊳ H. Then for l ∈ 1, 2, ..., n,* $(S \cap V_l) \cup (g_l, h_o)$ *is a geodetic set of G ⊳H*$[V_l]$.

2. Geodetic Number for Comb product of graph

Let $C_P = G \rhd H$. Let $V(G) = \{u_1, u_2, u_3, ..., u_m\}$ and $V(P_n)=\{v_1, v_2, v_3, ..., v_n\}$. Fromthe definition of the Comb product of graphs $V(G \rhd H) =(a, v)|a \in V(G)$, v ∈ V(H) and$(a, v)(b, w)\in E(G \rhd H)$ whenever $a = b$ and $vw\in E(H)$, or $ab\in E(G)$ and $v = w = o$.

We consider two different vertices $a, b \in V(G)$ and a vertex $o \in V(H)$. We define $H(a) = (a, v)|v \in V(H)$, $G(o) = (v, o)|v \in V(G)$, and $P_G(a, b)$ is the shortest path from a to b in G.

Lemma2.1. *Let* $C_P = G \rhd H$ *and let* $S \in V(G)$*, if any vertex of S is identifying inV(H) then* $I_G[S] = V(G \rhd H)$

Proof. Let $C_P = G \rhd H$ and let $S \in V(G)$, let us assume that $\{u, v, w\} \in S$ by taking the comb product of $V(G \rhd H) = (a, v)|a \in V (G), v \in V(H)$ and $(a,v)(b,w) \in E(G \rhd H)$ whenever $a = b$ and $vw\in E(H)$, or $ab \in E(G)$ and v=w=o. We consider two different vertices $a, b \in V(G)$ and a vertex $o \in V(H)$. We define $H(a) = (a, v)|v \in V (H)$, $G(o) = (v, o)|v \in V(G)$, and $P_G(a, b)$ is the shortest path from a to b in G. So that $V(H)$ is an interior vertex of a shortest path between a vertex of S.

We obtain four main results. The first result is related to $G \rhd H$when G is a connected graph and H is the family of graphs such as Path, Cycle, Completed Graph, Trees and Wheel Graph.

3. Geodetic Number for Comb product of graph Gand H

Theorem3.1. $g(G \rhd P_n) = m$ *where* $m, n \geq 2$

Proof. Let $C_P = G \rhd P_n$ and let $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(P_n)=\{v_1, v_2, ..., v_n\}$,let us assume that v_i and v_j be a pendent vertices. Then we know that $S = \{v_i, v_j\}$. Bythe definition of comb product we can identifying $V(G)$ copies of H. Let us take v_i is anidentifying vertex of all the copies. Then the resultant graph gives $V(G)$ copies of pendentvertices. Already we know that all the pendent vertices should be belongs to a geodetic set.By the Lemma 2.1 $V(G)$ copies of pendent vertices is geodetic closure of $V(P_n)$.

Therefore$|S| = m$

Theorem3.2. $g(G \rhd C_n) = \begin{cases} m & if\ n\ is\ even \\ m+1 & if\ n\ is\ odd \end{cases}$, *where* $m, n \geq 3$

Proof. Let $C_P = G \rhd C_n$and let $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(C_n) = \{v_1, v_2, ..., v_n\}$

Case i: If *n* is even

Let us assume that H is an even cycle then the geodetic set of H is S={v_1, $v_{\frac{n}{2}}$}. Bythe definition of comb product we can identifying $V(G)$ copies of C_n. Let us take v_1 is anidentifying vertex of all the copies. Then the resultant graph gives $V(G)$ copies of $v_{\frac{n}{2}}$ vertexin $G \rhd C_n$. Already we know that vertex $v_{\frac{n}{2}}$ is center of the graph $C_n$so that the shortest pathbetween the $V(G)$ copies of $v_{\frac{n}{2}}$ is geodetic closure of $V(C_n)$. Therefore $|S| \leq m$.

We claim that $|S| \geq m$, Suppose We assume that $\{S - v_{\frac{n}{2}}\}$ is geodetic set in$G \rhd C_n$, and $v_{\frac{n}{2}} \in i^{th}$ copy $V(G)$ then that i^{th} copy intermediate vertices are does not belongsto the geodetic closure $I_{G \rhd C_n}[S]$ by the definition of the geodetic set it is a contradiction. So$|S| \geq g(C_n).m - m$.

Therefore $g(G \rhd C_n) = m$

Case ii: If *n* is odd

Let us assume that H is an odd cycle then the geodetic set of H is$S=\{v_1, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\}$. Bythe definition of comb product we can identifying $V(G)$ copies of H. Let us take v_1 is anidentifying vertex of all the copies. Then the resultant graph gives $V(G)$ copies of $\{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\}$vertex in $G \rhd C_n$. Already we know

that vertex shortest path between $(v_{\frac{n}{2}},v_{\frac{n}{2}+1})$ geodeticcloser C_n, so that the shortest path between the $V(G)$ copies of $(v_{\frac{n}{2}},v_{\frac{n}{2}+1})$ is geodetic closureof $V(C_n)$. Therefore $|S| \leq m+1$
We claim that $|S| \geq m+1$, Suppose We assume that $\{S - v_{\frac{n}{2}}\}$ is geodetic set in$G \rhd C_n$, and $v_{\frac{n}{2}} \in i^{th}$ copy $V(G)$ then that i^{th} copy intermediate vertices are does not belongsto the geodetic closure $I_{G \rhd C_n}[S]$, similarlyfor$\{S - v_{\frac{n}{2}+1}\}$thereforeby the definition of the geodetic set it is a contradiction. So$|S| \geq m+1$. Therefore $g(G \rhd C_n) = m+1$

Theorem3.3. *$g(G \rhd K_n) = m.g(K_n) - m$, where $m, n \geq 3$*
Proof. Let $C_P = G \rhd K_n$and let $V(G)=\{u_1,u_2,..., u_m\}$ and $V(K_n)=\{v_1,v_2,..., v_n\}$,we know that $S \in V(G)$, $S = \{v_1, v_2, ..., v_n\}$ By the definition of comb product we canidentifying $V(G)$ copies of K_n. Let us take v_1 is an identifying vertex of all the copies. Thenthe resultant graph gives $V(G)$ copies of $\{S/v_1\}$. Already we know that distance betweenany vertexes is 1 so that the Lemma 2.1 $V(G)$ copies of $\{S/v_1\}$ vertices is geodetic closure of$V(K_n)$.
Therefore $|S| \leq g(K_n).m - m$.We claim that $|S| \geq g(K_n).m - m$, Suppose We assume that $\{S - v_k\}$ is geodetic set in$G \rhd K_n$, and $v_k \in i^{th}$ copy $V(G)$ then that i^{th} copy intermediate vertices are does not belongsto the geodetic closure $I_{G \rhd K_n}[S]$ by the definition of the geodetic set it is a contradiction. So$|S| \geq g(K_n).m - m$. Therefore $g(G \rhd K_n) = m.g(K_n) - m$

Theorem3.4. *$g(G \rhd W_{1,n})= m.g(W_{1,n}) - m$, where $m, n \geq 4$*
Proof. Let $C_P = G \rhd W_{1,n}$ and let $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(W_{1,n}) = \{v_1, v_2, ..., v_n\}$,we know that $S \in V(G)$, $S = \{v_1, v_3, v_5..., v_{n-1}\}$ By the definition of comb product we canidentifying $V(G)$ copies of $W_{1,n}$. Let us take v_1 is an identifying vertex of all the copies. Thenthe resultant graph gives $V(G)$ copies of $\{S/v_1\}$. Already we know that in the Wheel graph distance betweenany vertex is 2 so that the Lemma 2.1 $V(G)$ copies of $\{S/v_1\}$ vertices is geodetic closure ofV$(W_{1,n})$. Therefore $|S| \leq g(W_{1,n}).m - m$
We claim that $|S| \geq g(W_{1,n}).m - m$, Suppose We assume that $k \in S$ina geodetic set. Consider $\{S - k\}$ is geodetic set in$G \rhd W_{1,n}$, and $k \in i^{th}$ copy $V(G)$ then that i^{th} copy intermediate vertices are does not belongsto the geodetic closure $I_{G \rhd W_{1,n}}[S]$.By the definition of the geodetic set it is a contradiction. So$|S| \geq g(W_{1,n}).m - m$
Therefore $g(G \rhd W_{1,n}) = m.g(W_{1,n}) - m$

Theorem3.5. *$g(G \rhd T_n)= m.g(p_k) - m$, where $m, n \geq 4$ and $p_k$is a set of all pendent vertices of T_n*
Proof. Let $C_P = G \rhd T_n$ and let $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(T_n) = \{v_1, v_2, ..., v_n\}$,we know that $S \in V(G)$, $S = \{p_1,p_2,p_3,... \}$. Where $\{p_1,p_2,p_3,...\}$.Pendent vertices of the Tree. By the definition of comb product we can identifying $V(G)$ copies of T_n. Let us take v_1 is an identifying vertex of all the copies. Thenthe resultant graph gives $V(G)$ copies of $\{S/v_1\}$. Already we know that in a tree there exist unique path between any pair of vertices then the Lemma 2.1 $V(G)$ copies of $\{S/v_1\}$ vertices is geodetic closure of V(T_n). Therefore $|S| \leq g(p_k).m - m$
We claim that $|S| \geq g(p_k).m - m$, Suppose We assume that $p_{k-1} \in S$ina geodetic set. Consider $\{S - p_{k-1}\}$ is geodetic set in$G \rhd T_n$, and $p_{k-1} \in i^{th}$ copy $V(G)$ then that i^{th} copy intermediate vertices are does not belongsto the geodetic closure $I_{G \rhd T_n}[S]$.By the definition of the geodetic set it is a contradiction. So$|S| \geq g(p_k).m - m$
Therefore $g(G \rhd T_n)= m.g(p_k) - m$.
From the previous theorems we obtained the following corollary for Comb Product of any graph G and H

Corollary 3.6. *$g(G \rhd H) = m.g(H) - m$, where $m, n \geq 2$*

References
[1] M. Azari and A. Iranmanesh, Chemical graphs constructed from rooted product and their Zagreb indices, MATCH Commun. Math. Comput. Chem. 70 (2013), 901–919.
[2] B. Bresar, S. Klavzar and A. Douglas F. Rall, Dominating direct product of graphs, Discrete Math. 307 (2007) 1636-1642.
[3] B. Bresar, S. Klavzar and A. TepehHorvat, On the geodetic number and related metricsets in

Cartesian product graphs, Discrete Math. 308 (2008) 5555-5561.
[4] B. Bresar, S. Klavzar, T. K. Sumenjak and A. TepehHorvat, The geodetic number of lexicographic product of graphs, Discrete Math. 311 (2011) 1693-1698.
[5] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, On the geodetic and the hull numbers in strong product graphs, Comput. Math. Appl. 60 (2010) 3020-3031.
[6] G. Chartrand, F. Harary and P. Zhang, On the geodetic number of a graph, Diss. Math.Graph theory 20 (2000) 129 - 138.
[7] G. Chartrand, F. Harary and P. Zhang, Geodetic sets in graph, Networks 39 (2002)1-6.
[8] G. Chartrand and L. Lesniak (2005), Graphs and Digraphs, Fourth Edition, CRC Press,Boca Raton.
[9] Dimas AgusFahrudin, SuhadiWidoSaputro (2020), The geodetic-dominating numberof comb product graphs, Electronic Journal of Graph Theory and Applications 8 (2)(2020), 373–381.
[10] F. Harary, E. Loukakis and C. Tsouros, The geodetic number of a graph, Math. Comput.Modelling 17 (1993) 89-95.
[11]Saputro, S.W. &Mardiana, N. &Purwasih, I.A.. (2017). The metric dimension of comb product graphs. MatematickiVesnik. 69. 248-258.
[12] T. Jiang, I. Pelayo and D. Pritikin, Geodesic convexity and Cartesian products in graphs, manuscript, 2004.
[13] W. Imrich and S. Klavzar. Products graphs: Structure and Recognition. Wiley-Interscience, New York, 2000.